

A note on effective $N = 1$ super-Yang–Mills theories versus lattice results[★]

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Abstract. We compare the glueball mass spectrum of an effective $N = 1$ pure super Yang–Mills theory formulated in terms of a three-form supermultiplet with the available lattice data. These confirm the presence of four scalars and two Majorana fermions but the detailed mass spectrum is difficult to reconcile with the effective supersymmetric theory. By imposing supersymmetry and using two of four bosonic masses we get a prediction for the remaining masses as well as the mixing angles. We find that the mass of the three-form dominates over the contribution of the Veneziano–Yankielowicz–Dijkgraaf–Vafa term. As a byproduct we introduce a Fayet–Iliopoulos term for the three-form multiplet and show that it generates a glueball condensate.

1 Introduction

Recently there has been considerable progress in understanding some aspects of strongly coupled $\mathcal{N} = 1$ supersymmetric gauge theories in four space-time dimensions below their confinement scale [1–3]. More precisely, corrections to the Veneziano–Yankielowicz superpotential [4] have been proposed and conjectured to give an exact superpotential $W(S)$ in terms of the glueball superfield $S = \text{tr } W^\alpha W_\alpha$. Among other things these developments strengthened the belief that an effective action is the appropriate description of confined supersymmetric $\mathcal{N} = 1$ gauge theories.

However, it has been pointed out in [5] that the Veneziano–Yankielowicz superpotential only gives rise to mass terms of the (complex) gluino condensate $\langle \lambda\lambda \rangle$ but that the glueballs $\langle F_{\mu\nu} F^{\mu\nu} \rangle$ and $\langle F_{\mu\nu} \tilde{F}^{\mu\nu} \rangle$ remain massless. The reason is that in the Veneziano–Yankielowicz approach the glueball $\langle F_{\mu\nu} F^{\mu\nu} \rangle$ appears in the auxiliary F component of the chiral superfield S and hence no mass term can arise. S only contains two physical scalars and therefore cannot be adequate to describe the dynamics of the four bound states $\langle \lambda\lambda \rangle$, $\langle \bar{\lambda}\bar{\lambda} \rangle$, $\langle F_{\mu\nu} F^{\mu\nu} \rangle$ and $\langle F_{\mu\nu} \tilde{F}^{\mu\nu} \rangle$.

However, as stressed in [5, 6] S really is a constrained chiral multiplet and should better be viewed as the field

strength $S \sim \bar{D}^2 U$ of a three-form multiplet U . Adopting this point of view it is possible to add a supersymmetric mass term for U and in this way introduce two additional massive bosonic and fermionic degrees of freedom and generate glueball masses [5].

Independent of these developments lattice simulations of supersymmetric pure $SU(2)$ gauge theories have been improved [7–9].¹ Most of the lattice computations use Wilson-type lattice actions where supersymmetry is softly broken by a gluino mass term and later recovered in the continuum limit. The spectrum of the low-lying glueball and gluino condensates has been computed and shown to contain four scalar fields and two Majorana fermions. Furthermore, supersymmetric Ward identities have been checked indicating that supersymmetry is recovered in the continuum limit. The purpose of this letter is to compare the lattice results of [7] a little more carefully with the approach of [5] and show that in order to reach agreement strong consistency constraints for both the lattice simulations and the low energy effective Lagrangian emerge. We first briefly review the proposal of [5] and then compare it with the lattice simulations. By imposing supersymmetry and fitting two bosonic masses we get a prediction for the remaining masses as well as the mixing angles. We find that the mass of the three-form dominates over the contribution of the Veneziano–Yankielowicz–Dijkgraaf–Vafa term. As a byproduct we also introduce a Fayet–Iliopoulos term for the three-form multiplet and show that it generates a glueball condensate $\langle F_{\mu\nu} F^{\mu\nu} \rangle$.

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¹ The results on $SU(3)$ are in [10] while the subject is reviewed in [11]

2 The Veneziano–Yankielowicz effective action

The starting point is a pure $\mathcal{N} = 1$ supersymmetric $SU(N)$ gauge theory with a vector multiplet V in the adjoint representation of $SU(N)$. As physical components it includes a vector field v_μ and a gluino λ_α both in the adjoint representation of $SU(N)$. The superspace Lagrangian reads [12]

$$\begin{aligned} \mathcal{L} &= -\frac{i}{8\pi} \int d^2\theta \tau \operatorname{tr} W^\alpha W_\alpha + \text{h.c.} \\ &= -\frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} - \frac{\Theta}{16\pi^2} \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{i}{g^2} \operatorname{tr} \bar{\lambda} \bar{\sigma}^\mu D_\mu \lambda, \end{aligned} \quad (1)$$

where $\tau \equiv \frac{\Theta}{2\pi} + i\frac{4\pi}{g^2}$ is the complex gauge coupling and W_α is the superfield which contains the field strength $F_{\mu\nu}$. It is defined in terms of a real vector superfield $V = V^\dagger$ as

$$W_\alpha = -\frac{1}{4} \bar{D}^2 e^{-V} D_\alpha e^V, \quad (2)$$

Due to its definition it obeys [6, 12]

$$\bar{D}_{\dot{\alpha}} W_\alpha = 0, \quad \mathcal{D}^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}, \quad (3)$$

where $\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}$ are the gauge covariant superspace derivatives.

It is believed that this asymptotically free gauge theory confines below the scale

$$\Lambda = M e^{\frac{2\pi i \tau}{3N}}, \quad (4)$$

where M is some high energy scale at which the gauge theory (and the coupling τ) are defined. Below the confinement scale Λ colorless bound states form such as the gluino condensates $\langle \lambda^\alpha \lambda_\alpha \rangle, \langle \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \rangle$ and the CP -even and CP -odd glueballs $\langle F_{\mu\nu} F^{\mu\nu} \rangle, \langle F_{\mu\nu} \tilde{F}^{\mu\nu} \rangle$. They do not break supersymmetry [13] but they do break the chiral symmetry $\lambda \rightarrow e^{i\alpha} \lambda$ of the original theory (1).

Veneziano and Yankielowicz [4] proposed an effective description below the confinement scale Λ in terms of a chiral superfield $S := \operatorname{tr} W^\alpha W_\alpha$ with the effective Lagrangian²

$$\mathcal{L}_{\text{eff}} = \int d^4\theta K_{\text{eff}}(S, \bar{S}) + \int d^2\theta W_{\text{eff}}(S), \quad (5)$$

where

$$W_{\text{eff}}(S) = \hat{N} S \left(\ln \frac{S}{\Lambda^3} - 1 \right), \quad \hat{N} \equiv \frac{N}{32\pi^2}. \quad (6)$$

The superpotential W is designed to reproduce the chiral anomaly. K was originally fixed by dimensional analysis and superconformal anomalies to be $K_{\text{eff}}(S, \bar{S}) = \frac{1}{\alpha} (S \bar{S})^{\frac{1}{3}}$

² In [4, 5] a different definition is used: $S = \frac{\beta(g)}{2g} \operatorname{tr} W^\alpha W_\alpha$ where $\beta(g)$ is the (exact) β -function. Here we prefer to define S without factors of the gauge coupling in order to keep the holomorphic properties transparent. In string theory τ is not a constant but rather a dynamical chiral superfield

with α being a dimensionless normalization constant [4, 14]. However, by allowing K_{eff} to explicitly depend on Λ more general Kähler potentials are conceivable. Therefore in our analysis we will not use a specific K_{eff} but instead express everything in terms of appropriate derivatives of K_{eff} .

In accord with the Witten index [13] this effective theory has N supersymmetric ground states determined by $\frac{\partial W_{\text{eff}}}{\partial S} = 0$ which correspond to³

$$\langle S \rangle = \Lambda^3 e^{\frac{2\pi i n}{N}}, \quad n = 0, \dots, N-1. \quad (7)$$

The appearance of these N ground states from the minimization of W is a bit tricky and has been discussed in [3, 15]. We return to this issue in our discussion of the three-form multiplet.

Dijkgraaf and Vafa [1] added a chiral multiplet in the adjoint representation of $SU(N)$ to the original pure supersymmetric Yang–Mills theory (1). By giving this multiplet a large mass it can be integrated out of the effective action but it leaves behind polynomial corrections in W which are of the form $W_{\text{eff}}(S) = \hat{N} S (\ln \frac{S}{\Lambda^3} - 1) + \hat{N} \sum_n a_n S^n$. These corrections shift the location of $\langle S \rangle$ and also shift the mass term.

S has an expansion in terms of component fields:

$$S = \operatorname{tr} \lambda^\alpha \lambda_\alpha + \dots - \theta^2 \left(\operatorname{tr} \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} + \dots \right), \quad (8)$$

and thus (7) implies the formation of the gluino condensate $\langle \lambda^\alpha \lambda_\alpha \rangle$ while the glueball condensates $\langle F_{\mu\nu} F^{\mu\nu} \rangle$ and $\langle F_{\mu\nu} \tilde{F}^{\mu\nu} \rangle$ do not form. Similarly, expanding W_{eff} around $\langle S \rangle$ one finds a mass term for $\langle \lambda^\alpha \lambda_\alpha \rangle$ but no glueball masses.

It is this fact that led Farrar, Gabadaze and Schwetz to propose a modification of the VY effective action [5] by formulating the effective theory in terms of a three-form multiplet U .⁴ The necessity to amend or reformulate the VY description had been stressed before in [6, 15]. The common criticism amounts to the fact that the second constraint in (3) should be taken seriously as a quantum constraint. In terms of $S = \operatorname{tr} W^2$ this constraint reads

$$D^2 S - \bar{D}^2 \bar{S} = \Omega, \quad (9)$$

where Ω is a superfield whose lowest component is the topological density $\operatorname{tr} F \tilde{F}$, i.e.

$$\Omega = i\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \theta \dots = -4i\epsilon^{\mu\nu\rho\sigma} \partial_\mu \omega_{\nu\rho\sigma} + \theta \dots, \quad (10)$$

with $\omega_{\nu\rho\sigma} = \operatorname{tr} (v_\nu \partial_\rho v_\sigma - \frac{2}{3} i v_\nu v_\rho v_\sigma)$ being the Chern–Simons three-form. In other words, the lowest component

³ The $U(1)$ chiral symmetry is not completely broken by the anomaly but appropriate integer shifts of θ leave a discrete \mathbf{Z}_{2N} intact. The gluino condensate is only invariant under $\lambda \rightarrow -\lambda$ and thus it breaks the \mathbf{Z}_{2N} to \mathbf{Z}_2 . As a consequence N different ground states appear which are parameterized by the phase of S

⁴ Alternative ways of generating glueball masses have been suggested in [16]

of Ω is the field strength of a three-form. In fact S itself can be viewed as the field strength supermultiplet of a three-form multiplet U . Before we come to the effective action let us therefore briefly recall some facts about the three-form multiplet [6, 17].

3 The three-form multiplet

A real vector superfield V has in its $\theta\bar{\theta}$ component a vector field v_μ . However, one can equivalently use the Hodge dual three-index antisymmetric tensor or in other words the three-form C_3 as the $\theta\bar{\theta}$ component of a real vector superfield.⁵ The difference emerges when one considers the corresponding field strengths which are not dual to each other. The field strength of a vector superfield V is the chiral superfield W_α introduced in (2) which contains $F_{\mu\nu}$ as the θ component and is invariant under the gauge transformations $V \rightarrow V + \Phi + \bar{\Phi}$ where Φ is a chiral superfield. In components this transformation contains the standard gauge transformation $v_\mu \rightarrow v_\mu + \partial_\mu \alpha$.

Let $U = U^\dagger$ be the superfield which contains C_3 in its $\theta\bar{\theta}$ component. Its field strength S is defined by⁶

$$S = -4\bar{D}^2 U, \quad \bar{S} = -4D^2 U, \quad (11)$$

which is a constrained chiral superfield in that it satisfies

$$\bar{D}_\alpha S = 0, \quad D^2 S - \bar{D}^2 \bar{S} = \Omega. \quad (12)$$

Here Ω is a superfield which contains the four-form field strength $F_4 = dC_3$ in its lowest component. S (and Ω) are left invariant by the superfield gauge transformation $U \rightarrow U + L$, where L is a linear multiplet obeying $D^2 L = \bar{D}^2 L = 0$. At the component level this corresponds to the three-form gauge invariance $C_3 \rightarrow C_3 + d\Theta_2$ which leaves F_4 invariant. Note that (12) does allow for the possibility of a supersymmetric VEV $\langle S \rangle$ since (12) is invariant under the (supersymmetric) shift $S \rightarrow S + \text{const}$. Thus a more precise version of (11) reads $S = \langle S \rangle - 4\bar{D}^2 U$. We will come back to this issue in the next section.

A massless three-form contains no physical degree of freedom since its field strength F_4 is dual to a constant. At the level of superfields this duality states that a three-form multiplet is dual to a chiral multiplet with the physical degrees of freedom being a Weyl fermion and a complex scalar [6]. Since the gauge invariance is unbroken the most general action can be expressed in terms of only the field strength S and reads

$$\mathcal{L} = \int d^4\theta K(S, \bar{S}) + \int d^2\theta W(S) + \text{h.c.} \quad (13)$$

Thus we see that standard interactions of the chiral field strength S describe a massless three-form multiplet. However, the presence of the three-form does change the mini-

um energy condition. After carefully dualizing the three-form one finds that the potential is minimized by [6]

$$\frac{\partial \hat{W}(S)}{\partial S} = 0, \quad \text{where} \quad \hat{W}(S) = W(S) + icS. \quad (14)$$

At the tree level c is a real constant, the dual of F_4 . Adding a term icS to W has also been advocated in [3, 15] by a different reasoning. Here we see that it naturally appears if one takes S to be the field strength of a three-form multiplet. Furthermore, the effective theory is known to have domain wall solutions interpolating between the N different ground states of (7) [18]. The three-form C_3 is the gauge field which naturally couples to these domain walls. The charge satisfies a Dirac-type quantization condition which in turn results in the quantization $c = \frac{n}{16\pi}$, $n \in \mathbf{N}$ [18, 19]. Using (14) one now finds the N different ground states displayed in (7) as the minimum energy condition. So far we considered a massless three-form multiplet. Let us now discuss the modifications which appear when a mass term and a Fayet–Iliopoulos term for C_3 are included. A massive three-form has one physical degree of freedom which it gains by “eating” an appropriate Goldstone boson. This Goldstone boson is a two-form B_2 with an invariant coupling $\mathcal{L} \sim (C_3 - dB_2)^2$. B_2 can be removed from the Lagrangian by an appropriate gauge transformation. In a supersymmetric theory B_2 resides in a linear multiplet G and thus the Lagrangian (13) receives the additional terms

$$\delta\mathcal{L}_{m_U} = - \int d^4\theta \left(\frac{1}{2} m_U^2 (U - G)^2 - \xi(U - G) \right). \quad (15)$$

We see that the gauge invariance $U \rightarrow U + L$ can be maintained by assigning the transformation law $G \rightarrow G + L$ to the linear multiplet G . Keeping the three-form gauge invariance is crucial since for the $SU(N)$ gauge theory it is related to ordinary gauge invariance. This can be seen from the fact that in this case the three-form is nothing but the Chern–Simons three-form which does transform under the (non-Abelian) gauge symmetry. In this way the three-form gauge invariance is linked to $SU(N)$ invariance.

If one fixes a gauge $U = U' + G$ (“unitary gauge”) G disappears from the action or in other words it is “eaten” by U . In this gauge the “longitudinal” degrees of freedom of U which are a gauge redundancy in the massless case become physical degrees of freedom. One bosonic degree of freedom is represented by the massive three-form which is dual to a scalar. Supersymmetry requires that this scalar comes accompanied with an additional bosonic and two fermionic degrees of freedom originally residing in G . Thus, the massive three-form multiplet has altogether four bosonic and four fermionic degrees of freedom, i.e. twice the number of physical degrees of freedom of S .⁷ To see this more explicitly let us now turn to the effective action suggested in [5].

⁷ In [5] it is suggested that the massive three-form multiplet can be described equivalently by two chiral multiplets

⁵ The two fields are related via $v_\mu \sim \epsilon_{\mu\nu\rho\sigma} C^{\nu\rho\sigma}$

⁶ A generic chiral superfield Φ can always be written in terms of an unconstrained superfield X as $\Phi = \bar{D}^2 X$, $\bar{\Phi} = D^2 \bar{X}$, but in general X is not real

4 The effective action of a massive three-form

References [5] propose a modification of the VY effective action such that also glueball masses can be accommodated. Here we further generalize this action by also adding a Fayet–Iliopoulos term which will lead to the possibility of describing a glueball condensate. The basic idea is to take the constraints (3) seriously also at the quantum level and view S not as a chiral field but as a constrained chiral multiplet or in other words as the field strength of a three-form multiplet. In this case the basic variable of the effective theory is not S but rather U and the effective action should be formulated in terms of U . Adding (S -dependent) mass and Fayet–Iliopoulos terms one has⁸

$$\mathcal{L} = \int d^4\theta \left(K_{\text{eff}}(S, \bar{S}) - \frac{1}{2} m_U^2(S, \bar{S}) (U - G)^2 + \xi(S, \bar{S}) (U - G) \right) + \int d^2\theta W_{\text{eff}}(S) + \text{h.c.}, \quad (16)$$

where $W_{\text{eff}}(S)$ is as in (6) but should now be viewed as a function of U . $K_{\text{eff}}, m_U^2, \xi$ are arbitrary functions of S ; in order to determine the minimum and the mass spectrum of the theory we do not really need to know their full analytic structure but only the first term in a Taylor expansion around $\langle S \rangle$. As we have already stated above, in order to accommodate a supersymmetric VEV for S the relation (11) has to be modified to $S = \langle S \rangle - 4\bar{D}^2 U$. Strictly speaking we should use this form of S to derive the effective action in components and then determine $\langle S \rangle$ by the minimum energy condition. As expected this procedure leads again to (7). In order not to overload the notation and to make the following formulas look more canonical let us chose the specific vacuum $\langle S \rangle = \Lambda^3$ right from the beginning and define

$$S = \Lambda^3 + \Lambda^2 \hat{S}, \quad \hat{S} \equiv -4\bar{D}^2 U, \quad \langle \hat{S} \rangle = 0. \quad (17)$$

Let us stress that choosing one of the other vacua of (7) yields an entirely equivalent result. In fact also for the massive three-form we recover the result of [6] that the minimum energy condition can be expressed as in (14). Using dimensional analysis we can constrain the leading terms in the Taylor expansion around $\langle S \rangle$ of the couplings to be

$$\begin{aligned} K_{\text{eff}}(S, \bar{S}) &= k \hat{S} \bar{\hat{S}} + \mathcal{O}((\hat{S} \bar{\hat{S}})^2), \\ m_U^2(S, \bar{S}) &= m_U^2 + \mathcal{O}(\hat{S} \bar{\hat{S}}), \\ \xi(S, \bar{S}) &= \xi \Lambda^2 + \mathcal{O}(\hat{S} \bar{\hat{S}}), \end{aligned} \quad (18)$$

where by slight abuse of notation k, m_U^2, ξ now denote constants.

The next task is to compute the Lagrangian (16) in components and determine the vacuum and the mass matrices. To a large extent this has already been done in [5]

⁸ Obviously, one can add higher powers of $U - G$ to \mathcal{L} . However, such terms do not influence the mass spectrum but correspond instead to additional interactions

and thus we can be very brief in the following. The only difference compared to [5] is that we do not use a specific S -dependence for the couplings K_{eff} and m_U^2 since they do not enter the mass matrices. Furthermore, we add a Fayet–Iliopoulos term in order to allow for the possibility of a non-trivial $\langle U \rangle$. Following [5] we expand U in component fields as follows

$$\begin{aligned} U &= \langle B \rangle + B + i\theta\chi - i\bar{\theta}\bar{\chi} + \frac{1}{16}\theta^2\bar{A} + \frac{1}{16}\bar{\theta}^2 A \\ &+ \frac{1}{48}\theta\sigma^\mu\bar{\theta}\epsilon_{\mu\nu\rho\kappa}C^{\nu\rho\kappa} + \frac{1}{2}\theta^2\bar{\theta}\left(\frac{\sqrt{2}}{8}\bar{\psi} + \bar{\sigma}^\mu\partial_\mu\chi\right) \\ &+ \frac{1}{2}\bar{\theta}^2\theta\left(\frac{\sqrt{2}}{8}\psi - \sigma^\mu\partial_\mu\bar{\chi}\right) + \frac{1}{4}\theta^2\bar{\theta}^2\left(\frac{1}{4}\Sigma - \partial^\mu\partial_\mu B\right), \end{aligned} \quad (19)$$

where B is a real and A a complex scalar field, $C^{\nu\rho\kappa}$ is the three-form, χ, ψ are Weyl fermions and Σ is an auxiliary field. We have already anticipated the fact that the lowest component of U will receive a VEV due to the presence of the Fayet–Iliopoulos term and therefore included a term $\langle B \rangle$. Using (17) one identifies the components of \hat{S} by

$$\hat{S} = A + \sqrt{2}\theta\psi + \theta^2(\Sigma + iF_4), \quad (20)$$

where $F_4 = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\partial^\mu C^{\nu\rho\sigma}$. In order to have canonically normalized kinetic terms for all fields we need to further rescale $\hat{S} \rightarrow k^{-\frac{1}{2}}\hat{S}$ and $B \rightarrow \frac{\sqrt{2}}{m_U}B$, $\chi \rightarrow \frac{\sqrt{2}}{m_U}\chi$. Inserted into (16) and using (18), (19) and (20) one arrives at a Lagrangian written in terms of the massive three-form C_3 and its field strength F_4 . The auxiliary field Σ is eliminated by its equation of motion,

$$\Sigma = \frac{m_U}{16\sqrt{2}k}B - \frac{\hat{N}\Lambda}{k}\text{Re}A. \quad (21)$$

Finally, dualizing C_3 to a scalar σ via

$$C_{\mu\nu\rho} = -\frac{16}{m_U}\sqrt{\frac{k}{2}}\epsilon_{\mu\nu\rho\kappa}\partial^\kappa\sigma, \quad (22)$$

we arrive at

$$\begin{aligned} \mathcal{L} &= -\partial_\mu\Phi^i\partial^\mu\Phi^i - i\bar{\Psi}^i\bar{\sigma}^\mu\partial_\mu\Psi^i - m_{ij}^2\Phi^i\bar{\Phi}^j \\ &- \left(\frac{1}{2}m_{ij}\Psi^i\Psi^j + \text{h.c.}\right) + \text{higher order interactions}, \end{aligned} \quad (23)$$

where $\Phi^i \equiv \left(A, \frac{1}{\sqrt{2}}(B + i\sigma)\right)$, $\Psi^i \equiv (\psi, \chi)$, $i = 1, 2$ and

$$m_{ij} = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & 0 \end{pmatrix}, \quad m_{11} = \frac{\hat{N}\Lambda}{k}, \quad m_{12} = \frac{m_U}{16\sqrt{k}}. \quad (24)$$

(Of course this is exactly the same result obtained in [5] which can be explicitly seen by using the correspondence $k = \frac{1}{9\alpha}$, $m_U = \sqrt{2/\delta}\Lambda$, $\hat{N} = \gamma$.)

Furthermore, a proper minimum of the potential requires

$$\langle B \rangle = \xi \frac{\Lambda^2}{m_U^2}. \quad (25)$$

As anticipated the Fayet–Iliopoulos term ξ induces a VEV for B . In terms of the original $SU(N)$ gauge theory B is a mixture of $\text{tr } F^2$ and $\lambda\lambda$ as can be seen from (8), (20) and (21).

Let us now turn to a discussion of the mass spectrum. From (24) we see that for the fermion ψ sitting in \hat{S} a Majorana mass term arises directly from the VY superpotential. m_U on the other hand induces a Dirac mass term for ψ – χ while no Majorana mass term arises for χ . In the next section we need the eigenvalues of the fermion mass matrix m_{ij} and the bosonic mass matrix m_{ij}^2 which are given by

$$m_f = \frac{1}{2}m_{11} \pm \sqrt{m_{12}^2 + \frac{1}{4}m_{11}^2},$$

$$m_b^2 = \frac{1}{2}m_{11}^2 + m_{12}^2 \pm m_{11}\sqrt{m_{12}^2 + \frac{1}{4}m_{11}^2}. \quad (26)$$

If we consider the correction to the superpotential W computed in [1] the VEV for S is shifted and an additional contribution to m_{11} arises. For instance, after the inclusion of a purely quadratic correction $\hat{N}a_2S^2$ to the superpotential, and parameterizing the new vacuum expectation value of S by $\langle S \rangle = \Lambda^3 + \Lambda^2\delta$, m_{11} is shifted according to

$$m_{11} \rightarrow m_{11}\Lambda \left(\frac{1}{\Lambda + \delta} + 2a_2\Lambda^2 \right), \quad (27)$$

while m_{12} remains unchanged.

5 Comparison with lattice results

Various simulations for pure $SU(2)$ super-Yang–Mills theories have been performed on the lattice [7]. The two basic issues arising are on the one hand the recovery of supersymmetry in the continuum limit and on the other hand the necessity to include dynamical chiral fermions. Most of the available lattice results use Wilson-type lattice actions with a bare gluino mass term added which breaks supersymmetry (and the chiral symmetry) softly. The gluino mass is then tuned such that supersymmetry is recovered in the continuum limit. Restoration of supersymmetry is checked by computing superconformal Ward identities.

The lattice simulations of [7] show a non-trivial mass spectrum for four scalar degrees of freedom and two Majorana fermions which seem to assemble in two chiral multiplets near the supersymmetric limit. Let us focus on the bosonic states. The lightest states are the CP -even glueball B (called 0^+ in [7]) and the CP -odd gluino condensate $\text{Im}A$ (called $a - \eta'$ in [7]) and they are almost degenerate in mass. The CP -odd glueball σ (0^-) and the CP -even gluino condensate $\text{Re}A$ ($a - f_0$) are also degenerate in mass and heavier than B and $\text{Im}A$. Furthermore, the mass difference between these two sets of states is much larger than the value of the gluino mass used in the simulation and therefore does not appear to be an effect of the softly broken supersymmetry.

However, taken at face value this is in conflict with basic properties of supersymmetry. From (23) we learn that the four bosonic states combine into two complex scalar fields A and $\frac{1}{\sqrt{2}}(B + i\sigma)$. In terms of the original $SU(N)$ gauge theory A corresponds to the gluino condensate $\text{tr } \lambda\lambda$ while $B + i\sigma$ is a mixture of $\text{tr } F^2 + i \text{tr } F\tilde{F}$ with A (c.f. (8), (20) and (21)). The two complex scalar fields A and $\frac{1}{\sqrt{2}}(B + i\sigma)$ do mix via the mass matrix (24) and therefore the mass eigenstates are a linear combination of A and $B + i\sigma$ with generically different masses (c.f. (26)). However, the mixing is only among complex scalar fields as can be seen from (23) in that no terms Φ^2 or $\bar{\Phi}^2$ appear. Therefore the CP -even and CP -odd states of the same complex scalar continue to be degenerate in mass. It is in fact a fundamental property of unbroken supersymmetry that CP -even and CP -odd states in the same multiplet have to be degenerate in mass.⁹ The supermultiplets can mix but the mass eigenstates again have to be supersymmetric multiplets and therefore the CP -even and CP -odd states of the same multiplet are mass degenerate. Thus, A can mix with $B + i\sigma$ but the resulting mass eigenstates cannot lead to a mass split between $\text{Im}A$ and $\text{Re}A$. In this respect the lattice results which show B and $\text{Im}A$ mass degenerate seem to be in conflict with unbroken supersymmetry.

There appear to be various possible resolutions of this puzzle. First of all it could be that due to the mixing the states in the lattice simulations have been misidentified. In the lattice simulation correlation functions of operators with given quantum numbers are computed for large Euclidean times where they are dominated by the lightest states with these quantum numbers. This can be used to extract the mass of the states. Thus, when computing the correlation function of, say, $\text{tr } F^2$, it is in principle possible that this correlation function is dominated by the CP -even gluino-ball $a - f_0$. This would imply a mixing between the bosonic states with the same parity which, however, is not observed in the lattice simulations [7]. Furthermore, experience from QCD appears to make this possibility unlikely [20].

The second possibility is that in the supersymmetric limit all four states are really degenerate in mass and that the 0^+ glueball B and the CP -odd gluino condensate $a - \eta'$ ($\text{Im}A$) are really in different but almost degenerate supermultiplets. Since the more reliable lattice measurements are the two light states this would mean that the masses of the heavy states have to considerably decrease as one comes closer to the supersymmetric limit. However, improved lattice simulations currently under way do not indicate any tendency in this direction [20].

Thus there remains a puzzle when comparing the mass spectrum obtained in lattice simulation with computations based on supersymmetric effective actions. The fact that the conflict between the two approaches is at such a fundamental level makes this only the more interesting. Estimati-

⁹ This is related to the R -symmetry of the super-algebra which preserves the complex structure and pairs CP -even and CP -odd scalars into a complex scalar field

ing finite size effects both on the lattice and analytically might shed some light on this puzzle.

Let us now take the second solution seriously and ask what we can learn for the mass spectrum if we insist on supersymmetry and only take the two low-lying states into account. In this case all states have to be almost degenerate and we can determine the parameters k and m_U . This “predicts” the masses for the 0^- and $a - f_0$ as well as their mixing angle. From (26) we learn that a degeneracy among all four bosonic and fermionic states is not so easy to achieve and requires

$$m_{12} \gg m_{11} \quad \Rightarrow \quad m_U \gg \frac{\hat{N}}{\sqrt{k}} \Lambda. \quad (28)$$

Thus the mass of the three-form m_U has to be the dominant contribution in the mass matrices and is far more important than the contribution arising from the Veneziano–Yankielowicz–Dijkgraaf–Vafa superpotential. This also implies that once the Dijkgraaf–Vafa correction is taken into account m_{11} given in (27) cannot be large. Furthermore, from (24) we see that in this limit the mixing angle of the bosons is minimal while the fermions mix maximally.

6 Conclusions

In this letter we expanded on a suggestion put forward in [5, 6] to reformulate the effective action of strongly coupled supersymmetric gauge theories in terms of a (massive) three-form multiplet in order to account for glueball masses. A gauge invariant formulation of the mass term requires the presence of additional degrees of freedom and doubles the spectrum compared to the original Veneziano–Yankielowicz effective theory. Indeed the lattice simulations do measure four massive scalars and two Majorana fermions and in this sense confirm the proposal of [5]. However, taken at face value the observed mass spectrum is incompatible with supersymmetry in that the CP -even and CP -odd part of the complex scalars show different masses. We discussed two possible resolutions of this puzzle. Either there is a misidentification of states or all states have to be almost degenerate in mass. This in turn requires that the mass term of the three-form is the dominant contribution and no mixing among the bosonic states occurs while the fermions have to be maximally mixed. It would be worthwhile to further improve the lattice simulation and shed light on this puzzle.

We also introduced a Fayet–Iliopoulos term ξ into the theory and showed that it can lead to a non-trivial glueball condensate $\langle \text{tr } F^2 \rangle \neq 0$. It would also be nice to measure $\langle \lambda \lambda \rangle$ and $\langle \text{tr } F^2 \rangle$ on the lattice and determine in this way Λ and ξ .

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